ZIV-ZAKAI BOUNDS FOR TIME DELAY ESTIMATION WITH FREQUENCY HOPPING AND MULTICARRIER SIGNALS IN WIDEBAND RANDOM CHANNELS

Brian M. Sadler

Army Research Laboratory
Adelphi, MD 20783
bsadler@arl.army.mil

Ning Liu, Zhengyuan Xu

Department of Electrical Engineering
University of California
Riverside, CA 92521
dxu@ee.ucr.edu

ABSTRACT

We develop Ziv-Zakai bounds (ZZBs) on time delay estimation (TDE) for known frequency hopping or multicarrier waveforms over random wideband (frequency selective) Gaussian channels, with a uniform prior on the delay. The channel model incorporates correlation, for example between subcarriers. A unified signal model is applicable to several cases of interest. The receiver does not have channel state information to estimate the time delay, but does have knowledge of the channel statistics. The ZZB provides a tight bound on the mean square error for Bayesian estimation in wideband random channels, revealing SNR threshold behavior indicating estimator breakdown.

Index Terms— Time delay estimation, Ziv-Zakai bound, multicarrier, frequency hopping, wideband channel, frequency-selective fading.

1. INTRODUCTION

Time delay estimation (TDE) is fundamental for many applications, including ranging, synchronization, communications, geolocation, and array processing. Various performance bounds have been developed for TDE performance [1], and the Ziv-Zakai bound (ZZB) [2] is among the best mean-square-error (MSE) bounds for predicting optimal estimation performance over a wide range of signal-to-noise ratios (SNRs); e.g., see Van Trees and Bell [3].

In many cases the channel is wideband (frequency-selective), random, and unknown [4]. In this work we develop a ZZB for TDE over wideband random channels, and consider multicarrier and frequency hopping signals. We assume that the receiver knows the random channel distribution, but does not know the channel realization. Here, both the multipath channel and noise are modeled as Gaussian random variables, and the channel coefficients across multiple frequencies may be correlated. Thus the resulting bound reveals TDE statistical behavior in wideband fading channels, when the receiver does not have prior knowledge of the channel, for two common wideband waveforms that provide frequency diversity. This generalizes previous work that assumes perfect channel knowledge, and employs a real-valued single carrier signal model [5]. Other related narrowband channel cases with diversity includes TDE of frequency hopping waveforms after synchronous dehopping [6], and parallel independent narrowband fading channels [7].

The log-likelihood ratio (LLR) for the associated hypothesis test in the ZZB derivation is shown to follow a general quadratic form of a complex Gaussian random vector. We then find the probability density function (pdf) of the LLR via a moment generating function (MGF) approach, that in turn leads to the minimum detection error probability expression needed to complete the ZZB derivation. An example shows typical ZZB behavior.

This work was supported in part by the Army Research Laboratory CTA on Communications and Networks under grant DAAD19-01-2-0011.

2. SIGNAL AND CHANNEL MODELS

We consider both frequency hopping (FH) and multicarrier (MC) waveforms in a unified model, using a basic pulse $p(t)$ that is modulated with known symbols. We let $p(t)$ be a symmetrically truncated square-root raised-cosine (SRRC) pulse with unit energy. For the FH case, the pulse during the ith hop and kth symbol period is denoted $p_{i,k}(t) = p(t-(i-1)MT_s - kT_s)$, where T_s is the symbol duration, and each hop has $M = 2K+1$ symbols with hop dwell time duration MT_s. For the MC case, the modulated pulses of all N subcarriers are transmitted simultaneously, where the pulse for the ith subcarrier and kth symbol is given by $p_{i,k}(t) = p(t-kT_s)$. Therefore we can represent the complex envelope of the FH and MC waveforms in a unified model as

$$s_i(t) = \sum_{k=-K}^{K} a_{i,k} p_{i,k}(t). \tag{1}$$

We assume the symbols $a_{i,k}$ are taken from a complex constellation such as PSK or QAM, and are known to the receiver. For notational convenience, we stack all M symbols for the ith hop/subcarrier in a vector $\boldsymbol{a}_i$.

Let the transmitted symbol energy be E_{tx}. The ith hop/carrier signal $\sqrt{E_{\text{tx}}} s_i(t)$ propagates through a convolutive random channel modeled as a tapped delay line with fixed spacing T_t, given by [5], [8]

$$g_i(t) = \sqrt{G_i} \sum_{l=1}^{L} \alpha_{i,l} \delta(t-(l-1)T_t). \tag{2}$$

Here G_i is the gain factor of the ith hop/carrier, L is the total number of taps, and $\alpha_{i,l}$ is the gain for the lth tap in the ith hop/carrier. Let $\boldsymbol{\alpha}_i = [\alpha_{i,1}, \cdots, \alpha_{i,L}]^T$. We model $\boldsymbol{\alpha}_i$ as a complex Gaussian random vector with distribution denoted by $\mathcal{CN}(\boldsymbol{\mu}_i, \boldsymbol{V}_i)$, where $\boldsymbol{\mu}_i$ is the mean vector and $\boldsymbol{V}_i$ is the covariance matrix. Stacking $\boldsymbol{\alpha}_i$ over the N hops/carriers gives the total channel gain vector $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_1^T, \cdots, \boldsymbol{\alpha}_N^T]^T$, which follows a multidimensional complex Gaussian distribution $\mathcal{CN}(\boldsymbol{\mu}_\alpha, \boldsymbol{V})$, where $\boldsymbol{\mu}_\alpha = [\boldsymbol{\mu}_1^T, \cdots \boldsymbol{\mu}_N^T]^T$, and $\boldsymbol{V}$ is composed of $\boldsymbol{V}_{ij}$ as the ith-row and jth-column block, and $\boldsymbol{V}_{ii} = \boldsymbol{V}_i$ are diagonal blocks. The channel $\boldsymbol{\alpha}_i$ is assumed to have unit power such that $tr(\boldsymbol{\mu}_i \boldsymbol{\mu}_i^H + \boldsymbol{V}_i) = 1$, where tr is the trace operator. This complex-valued channel model can encompass a variety of scenarios such as correlated FIR taps, as well as correlation across frequency hops or subcarriers.

Define $E_{\mathrm{rx}i} \triangleq G_i E_{\mathrm{tx}}$ and denote the propagation delay as t_0. The received signal of the ith hop/carrier is

$$\begin{aligned} y_i(t) &= \sqrt{E_{\mathrm{rx}i}} \sum_{l=1}^{L} \alpha_{i,l} s_i(t-(l-1)T_t - t_0) + n_i(t) \\ &= \boldsymbol{\alpha}_i^T \boldsymbol{s}_i(t-t_0) + n_i(t), \end{aligned} \quad (3)$$

where

$$\boldsymbol{s}_i(t-t_0) = \sqrt{E_{\mathrm{rx}i}} [s_i(t-t_0), \cdots, s_i(t-(L-1)T_t - t_0)]^T,$$

and $n_i(t)$ is complex circularly symmetric white Gaussian noise with double sided spectral density $\sigma_n^2 = N_0$. The SNR of the ith hop/carrier is defined by $\xi_i = \frac{E_{\mathrm{rx}i}}{\sigma_n^2}$. Our goal is to develop a performance bound on estimation of the propagation delay t_0, based on the received multi-hop or multicarrier signal.

3. DEVELOPMENT OF THE ZIV-ZAKAI BOUND

The development of the ZZB links estimation of time delay t_0 to a hypothesis testing problem that discriminates a signal at two possible delays [2]. Let $\widehat{t_0}$ be a time delay estimate. For a received signal $h(t)$ at one of the two possible delays $h(t-a)$ or $h(t-a-\Delta)$, where $\Delta > 0$ and $a, a+\Delta \in [0, T]$, the hypothesis test is given by

$$\begin{aligned} &\text{Decide } H_0: \quad t_0 = a \qquad \text{if } |\widehat{t_0} - a| < |\widehat{t_0} - a - \Delta|, \\ &\text{Decide } H_1: \quad t_0 = a+\Delta \quad \text{if } |\widehat{t_0} - a| > |\widehat{t_0} - a - \Delta|. \end{aligned} \quad (4)$$

Denote the estimation error by $\epsilon = \widehat{t_0} - t_0$, and let $P_e(a, a+\Delta)$ be the minimal probability of error achieved by the optimum detection scheme in making the above decision. If the two hypothesized delays are equally likely to occur, then the estimation mean-square error (MSE) is lower bounded by [2]

$$\overline{\epsilon^2} \geq \frac{1}{T} \int_0^T \Delta(T-\Delta) P_e(\Delta) d\Delta. \quad (5)$$

Evaluation of the bound (5) relies on finding the minimal probability of error $P_e(\Delta)$, in our case without knowing the channel realization. We will find $P_e(\Delta)$ by evaluating the log-likelihood ratio (LLR) test as follows. We find the conditional distribution of the received signal, conditioned on the channel realization. This is then averaged over the channel distribution, yielding the unconditional distribution, reflecting the lack of knowledge of the channel at the receiver. The LLR is shown to depend quadratically on the received signal, and in Section 4 we use a moment generating function (MGF) approach to find an expression for the LLR distribution. Finally, $P_e(\Delta)$ is found using the LLR distribution.

3.1. Received Signal Distribution

From equation (3), the received signal can be written as

$$y_i(t) = \boldsymbol{\alpha}_i^T \boldsymbol{s}_{i,m} + n_i(t), \quad (6)$$

where $\boldsymbol{s}_{i,m} = \boldsymbol{s}_i(t - m\Delta)$ and m takes the value 0 or 1 corresponding to hypotheses H_0 or H_1, respectively. We assume the receiver observation window T_0 is much longer than the sum of T and the duration of the channel output waveform. Collecting $y_i(t)$ and $n_i(t)$ in vectors we have,

$$\boldsymbol{y}(t) = [y_1(t), \cdots, y_N(t)]^T, \quad \boldsymbol{n}(t) = [n_1(t), \cdots, n_N(t)]^T.$$

From (6) and the noise independence, the joint distribution of $\boldsymbol{y}(t)$ conditioned on the channel gain and time delay $m\Delta$ is given by [4, p. 192]

$$\begin{aligned} p(\boldsymbol{y}(t)|\boldsymbol{\alpha}_i, m\Delta) &= \mathcal{K} \exp \sum_{i=1}^{N} \left[-\frac{1}{\sigma_n^2} \int_{T_0} \left\| y_i(t) - \boldsymbol{\alpha}_i^T \boldsymbol{s}_{i,m} \right\|^2 dt \right] \\ &= \mathcal{K} \exp \left[-\frac{1}{\sigma_n^2} \sum_{i=1}^{N} \left(\boldsymbol{\alpha}_i^H \boldsymbol{S}_{00,i} \boldsymbol{\alpha}_i - 2\mathrm{Re}\left\{ \boldsymbol{r}_{m,i}^H \boldsymbol{\alpha}_i \right\} + I_{yi} \right) \right], \\ & \qquad t \in [\tau, \tau + T_0] \end{aligned} \quad (7)$$

where $[\tau, \tau+T_0]$ is the observation window starting from τ, $\mathcal{K}$ absorbs all the terms independent of $\boldsymbol{\alpha}_i$ and $m\Delta$, and

$$\boldsymbol{S}_{m_1 m_2, i} \triangleq \int_{T_0} \boldsymbol{s}_{m_1,i}^* \boldsymbol{s}_{m_2,i}^T dt,$$

$$\boldsymbol{r}_{m,i} \triangleq \int_{T_0} \boldsymbol{s}_{m,i}^* y_i(t) dt, \quad I_{yi} \triangleq \int_{T_0} |y_i(t)|^2 dt,$$

with $m, m_1, m_2 = \{0, 1\}$. For compact notation we define

$$\begin{aligned} \boldsymbol{S}_{m_1 m_2} &\triangleq \mathrm{diag}\{\boldsymbol{S}_{m_1 m_2, 1}, \cdots, \boldsymbol{S}_{m_1 m_2, N}\}, \\ \boldsymbol{r}_m &= [\boldsymbol{r}_{m,1}^T, \cdots, \boldsymbol{r}_{m,N}^T]^T, \quad I_y = \sum_{i=1}^{N} I_{y,i}, \end{aligned} \quad (8)$$

where "diag" denotes a block diagonal matrix, $\boldsymbol{S}_{mm} = \boldsymbol{S}_{00} = \boldsymbol{S}_{11}$ is a Hermitian Toeplitz matrix independent of m. Now (7) can be rewritten as

$$p(\boldsymbol{y}(t)|\boldsymbol{\alpha}, m\Delta) = \mathcal{K} \exp \left[-\frac{1}{\sigma_n^2} \left(\boldsymbol{\alpha}^H \boldsymbol{S}_{00} \boldsymbol{\alpha} - 2\mathrm{Re}\left\{ \boldsymbol{r}_m^H \boldsymbol{\alpha} \right\} + I_y \right) \right]. \quad (9)$$

The exponent of (9) has a quadratic form in the complex Gaussian random vector $\boldsymbol{\alpha}$. We can average this over the channel $\boldsymbol{\alpha}$ using (25) from the Appendix with $s = 1$, which yields

$$\begin{aligned} p(\boldsymbol{y}(t)|m\Delta) &= E_\alpha \left\{ p(\boldsymbol{y}(t)|\boldsymbol{\alpha}, m\Delta) \right\} \\ &= \mathcal{K} |\boldsymbol{X}|^{-1} \exp \left\{ \boldsymbol{r}_m^H \boldsymbol{W} \boldsymbol{r}_m + 2\mathrm{Re}\{\boldsymbol{h}^H \boldsymbol{r}_m\} + c \right\}, \end{aligned} \quad (10)$$

where

$$\boldsymbol{X} = \boldsymbol{I} + \frac{1}{\sigma_n^2} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{S}_{00} \boldsymbol{V}^{\frac{1}{2}}, \quad \boldsymbol{W} = \frac{1}{\sigma_n^4} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{X}^{-1} \boldsymbol{V}^{\frac{1}{2}},$$

$$\boldsymbol{h} = \frac{1}{\sigma_n^2} \left(\boldsymbol{I} - \frac{1}{\sigma_n^2} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{X}^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{S}_{00} \right) \boldsymbol{\mu}_\alpha = \boldsymbol{H} \boldsymbol{\mu}_\alpha,$$

$$c = \boldsymbol{\mu}_\alpha^H \left(\frac{1}{\sigma_n^4} \boldsymbol{S}_{00} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{X}^{-1} \boldsymbol{V}^{\frac{1}{2}} \boldsymbol{S}_{00} - \frac{1}{\sigma_n^2} \boldsymbol{S}_{00} \right) \boldsymbol{\mu}_\alpha - \frac{I_y}{\sigma_n^2}.$$

Suppressing the terms independent of m, $p(\boldsymbol{y}(t)|m\Delta)$ is proportional to

$$p(\boldsymbol{y}(t)|m\Delta) \propto \exp \left\{ \boldsymbol{r}_m^H \boldsymbol{W} \boldsymbol{r}_m + 2\mathrm{Re}\{\boldsymbol{h}^H \boldsymbol{r}_m\} \right\}. \quad (11)$$

Note that $\boldsymbol{h}$ becomes $\boldsymbol{0}$ if the channel has zero-mean, i.e., if $\boldsymbol{\mu}_\alpha = \boldsymbol{0}$.

3.2. Log-likelihood Ratio Test

Using the received signal distribution in (11), the log-likelihood ratio (LLR) to decide on hypothesis H_m $(m = 0, 1)$ is

$$\mathcal{L} \triangleq \ln \frac{p(\boldsymbol{y}(t)|0)}{p(\boldsymbol{y}(t)|\Delta)} = \boldsymbol{r}^H \boldsymbol{\Psi} \boldsymbol{r} + 2\mathrm{Re}\{\boldsymbol{g}^H \boldsymbol{r}\} \underset{H_1}{\overset{H_0}{\gtrless}} 0, \tag{12}$$

where $\boldsymbol{r} = [\boldsymbol{r}_0^H \ \boldsymbol{r}_1^H]^H$,

$$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{W} \end{bmatrix}, \ \boldsymbol{g} = \begin{bmatrix} \boldsymbol{h} \\ -\boldsymbol{h} \end{bmatrix} = \boldsymbol{G}\boldsymbol{\mu}_\alpha. \tag{13}$$

An error occurs if $\mathcal{L} < 0|m = 0$, or if $\mathcal{L} > 0|m = 1$. Letting $\mathcal{L}_m \triangleq \mathcal{L}|H_m$, and $\widetilde{\boldsymbol{r}}_m \triangleq \boldsymbol{r}|H_m$, (12) becomes

$$\mathcal{L}_m = \widetilde{\boldsymbol{r}}_m^H \boldsymbol{\Psi} \widetilde{\boldsymbol{r}}_m + 2\mathrm{Re}\{\boldsymbol{g}^H \widetilde{\boldsymbol{r}}_m\}, \tag{14}$$

and the hypothesis test minimum error probability is

$$P_e(\Delta) = \frac{1}{2}\Pr\{\mathcal{L}_0 < 0\} + \frac{1}{2}\Pr\{\mathcal{L}_1 > 0\}. \tag{15}$$

To find the probabilities in (15), in the next subsection we find the distribution of $\widetilde{\boldsymbol{r}}_m$, and then use it to derive the distribution of $\mathcal{L}_m$.

3.3. The Distribution of $\widetilde{\boldsymbol{r}}_m$

Denoting $\boldsymbol{z}_{m,i} \triangleq \int_{T_0} \boldsymbol{s}_{m,i}^* n_i(t)dt$, stacking into vector $\boldsymbol{z}_m$ for $m = 0, 1$, and using (8) and (6) conditioned on H_0, then $\boldsymbol{r}_{m,i}$ can be expressed as

$$\boldsymbol{r}_{m,i}|H_0 \quad = \quad \int_{T_0} \boldsymbol{s}_{m,i}^* y_i(t) = \boldsymbol{S}_{m0,i}\boldsymbol{\alpha}_i + \boldsymbol{z}_{m,i}, \tag{16}$$

and with concatentation we obtain $\boldsymbol{r}_m|H_0 = \boldsymbol{S}_{m0}\boldsymbol{\alpha} + \boldsymbol{z}_m$. Stacking $\boldsymbol{r}_0|H_0$ and $\boldsymbol{r}_1|H_0$ as in (13), we obtain

$$\widetilde{\boldsymbol{r}}_0 = \boldsymbol{r}|H_0 = \boldsymbol{R}_0\boldsymbol{\alpha} + \boldsymbol{z}, \ \ \boldsymbol{R}_0 = \begin{bmatrix} \boldsymbol{S}_{00} \\ \boldsymbol{S}_{10} \end{bmatrix}, \quad \boldsymbol{z} = \begin{bmatrix} \boldsymbol{z}_0 \\ \boldsymbol{z}_1 \end{bmatrix}. \tag{17}$$

Similarly, conditioned on H_1, we find

$$\widetilde{\boldsymbol{r}}_1 = \boldsymbol{r}|H_1 = \boldsymbol{R}_1\boldsymbol{\alpha} + \boldsymbol{z}, \ \ \boldsymbol{R}_1 = \begin{bmatrix} \boldsymbol{S}_{01} \\ \boldsymbol{S}_{00} \end{bmatrix}. \tag{18}$$

Therefore, under either hypothesis the data vectors $\widetilde{\boldsymbol{r}}_0$ or $\widetilde{\boldsymbol{r}}_1$ are linear combinations of Gaussian vectors $\boldsymbol{\alpha}$ and $\boldsymbol{z}$, so that $\widetilde{\boldsymbol{r}}_m$ follows a Gaussian distribution $\widetilde{\boldsymbol{r}}_m \sim \mathcal{N}(\boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m)$, where

$$\boldsymbol{\mu}_m = \boldsymbol{R}_m\boldsymbol{\mu}_\alpha, \quad \boldsymbol{\Sigma}_m = \left(\boldsymbol{R}_m\boldsymbol{V}\boldsymbol{R}_m^H + \boldsymbol{\Gamma}\right),$$

$$\boldsymbol{\Gamma} = E\{\boldsymbol{z}\boldsymbol{z}^H\} = \sigma_n^2 \begin{bmatrix} \boldsymbol{S}_{00} & \boldsymbol{S}_{01} \\ \boldsymbol{S}_{10} & \boldsymbol{S}_{00} \end{bmatrix} = \sigma_n^2 \begin{bmatrix} \boldsymbol{R}_0 & \boldsymbol{R}_1 \end{bmatrix}.$$

It is computationally difficult to directly evaluate the probabilities in (15) with the distribution of $\widetilde{\boldsymbol{r}}_m$. Next we use an MGF approach to find the distribution of $\mathcal{L}_m$.

4. EVALUATING THE ZZB WITH THE MGF APPROACH

From the previous section, the LLR $\mathcal{L}_m$ in (14) is a quadratic form of the Gaussian vector $\widetilde{\boldsymbol{r}}_m$. To find the probability distribution of $\mathcal{L}_m$, we first obtain its MGF (or characteristic function) and then apply the Fourier transform. The transform can be computed efficiently using an FFT. From that result, the distribution of the $\mathcal{L}_m$ is used to evaluate the error probabilities in (15) via 1-dimensional integration, and then the resulting $P_e(\Delta)$ is used to find the ZZB in (5).

By the definitions in (13), $\boldsymbol{g}^T\boldsymbol{\Psi}^{-1}\boldsymbol{g} = 0$. Using this, we can rewrite $\mathcal{L}_m$ in (14) as

$$\mathcal{L}_m = \widetilde{\boldsymbol{x}}_m^H \boldsymbol{\Psi} \widetilde{\boldsymbol{x}}_m, \qquad \text{where } \ \widetilde{\boldsymbol{x}}_m = \widetilde{\boldsymbol{r}}_m + \boldsymbol{\Psi}^{-1}\boldsymbol{g}. \tag{19}$$

Note that $\widetilde{\boldsymbol{x}}_m$ is Gaussian distributed with variance $\boldsymbol{\Sigma}_m$ and mean

$$\boldsymbol{\mu}_{xm} = \boldsymbol{\mu}_m + \boldsymbol{\Psi}^{-1}\boldsymbol{g} = (\boldsymbol{R}_m + \boldsymbol{\Psi}^{-1}\boldsymbol{G})\boldsymbol{\mu}_\alpha.$$

We introduce the zero mean Gaussian random vector $\boldsymbol{u}_m$, obtained from $\widetilde{\boldsymbol{x}}_m$ by the transformation

$$\widetilde{\boldsymbol{x}}_m = \boldsymbol{\Sigma}_m^{\frac{1}{2}}\boldsymbol{P}_m\boldsymbol{u}_m + \boldsymbol{\mu}_{xm} = \boldsymbol{\Sigma}_m^{\frac{1}{2}}\boldsymbol{P}_m(\boldsymbol{u}_m + \boldsymbol{b}_m), \tag{20}$$

so that the variance of $\boldsymbol{u}_m$ is the identity matrix $\boldsymbol{I}$, and the vector $\boldsymbol{b}_m$ is a linear transformation of channel mean $\boldsymbol{\mu}_\alpha$ given by

$$\boldsymbol{b}_m = \boldsymbol{P}_m^H\boldsymbol{\Sigma}_m^{-\frac{1}{2}}\boldsymbol{\mu}_{xm} = \boldsymbol{P}_m^H\boldsymbol{\Sigma}_m^{-\frac{1}{2}}(\boldsymbol{R}_m + \boldsymbol{\Psi}^{-1}\boldsymbol{G})\boldsymbol{\mu}_\alpha. \tag{21}$$

In this transformation, $\boldsymbol{P}_m$ is a unitary matrix in the eigendecomposition of the Hermitian matrix given by

$$\boldsymbol{\Sigma}_m^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\Sigma}_m^{\frac{1}{2}} = \boldsymbol{P}_m \mathrm{diag}\{\lambda_1, \cdots, \lambda_{2LN}\}\boldsymbol{P}_m^H = \boldsymbol{P}_m\boldsymbol{\Lambda}\boldsymbol{P}_m^H. \tag{22}$$

From this, the elements u_{mk} of $\boldsymbol{u}_m$ are independent Gaussian random variables, each with zero mean and unit variance. Applying (20) and noticing (22), (19) can be written as

$$\mathcal{L}_m = (\boldsymbol{u}_m + \boldsymbol{b}_m)^H\boldsymbol{\Lambda}(\boldsymbol{u}_m + \boldsymbol{b}_m) = \sum_{k=1}^{2LN} \lambda_k \left|u_{mk} + b_{mk}\right|^2, \tag{23}$$

where b_{mk} is the k-th element of $\boldsymbol{b}_m$. The MGF is now obtained by applying eq. (26) from the Appendix to (23), yielding

$$\Theta_m(s) = \prod_{k=1}^{2LN} (1 - s\lambda_k)^{-1} \exp\left\{\frac{s\lambda_k|b_{mk}|^2}{1 - s\lambda_k}\right\}. \tag{24}$$

In (24), each of the $2LN$ product factors stems from the MGF of a scaled noncentral Chi-square random variable with one degree of freedom [9]. This observation is consistent with (23), consisting of weighted sums of independent noncentral Chi-square random variables, where each summation term corresponds to a factor in (24).

5. AN EXAMPLE

We adopt the root MSE (RMSE) of the TDE as the performance metric and use the MGF approach to evaluate the ZZB. $p(t)$ is a square-root raised cosine (SRRC) pulse with roll-off factor $\beta = 0$ and mean-square bandwidth $B = 1/12$, and symmetrically truncated with symbol duration $T_s = 12$ for FH and NT_s for the MC case. The channel has tap spacing $T_t = 1$ and $L = 5$ taps. Note that we normalize time to T_t. The time delay has a uniform prior over $[0, 30]$; i.e., $T = 30$. The transmission duration is NMT_s, which is a cycle of N hops for FH or the corresponding time for MC transmission, where the number of transmitted symbols per hop/subcarrier is set to $M = 80/N$. The channel fading distribution of $\boldsymbol{\alpha}_i$ and

SNR$_i$ are the same for each hop/subcarrier. The statistics $\boldsymbol{\mu}_{\alpha i}$ and $\boldsymbol{V}_i$ are generated using the exponential power decay profile given in [5], with the mean of the first tap obtained from a Rician-K factor, and all other taps with zero mean. The channel correlation coefficient between two channel taps of two hop/subcarriers is modeled as $\rho_{i,j} = \rho^{|t_i - t_j|}(1+\sqrt{-1}\sigma 2\pi f_\delta)/(1+(\sigma 2\pi f_\delta)^2)$, based on the exponential temporal correlation model [4, p. 394] and the frequency correlation in Jakes model [8, (1.5-20),], in which $\rho = 0.8$ is the temporal correlation between two neighboring taps according to [4, p. 389], and $\sigma = 1$ is the root-mean-square value of the delay spread. We assume the frequency separation between every pair of neighbor hops/subcarriers is the same and set it to $f_0 = 1$.

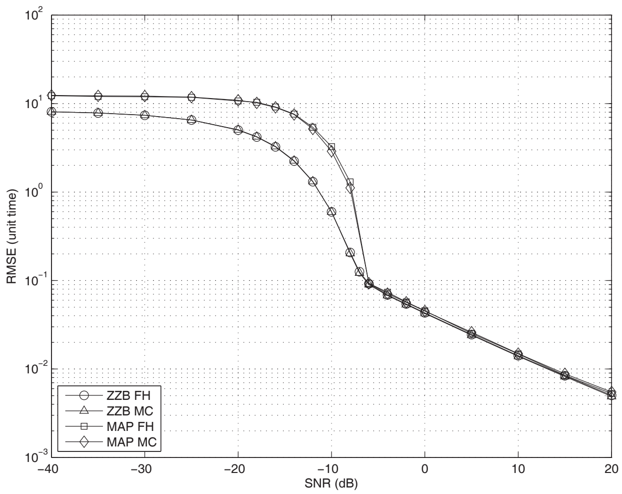

Fig. 1. ZZB and MAP estimation performance for equivalent FH and MC cases. $N = 4$, $M = 20$, and an FIR random channel, with first tap Rician-K = 20dB, and later taps Rayleigh distributed.

Figure 1 plots the ZZBs for both FH and MC waveforms in comparison with the performance of the optimum MAP estimator. The MAP estimator is implemented by using (10) (see also eq. (78) in [11]). The ZZBs predict typical behavior for TDE. At low SNR, the ZZB converges to $T/\sqrt{12}$, similar to other cases [5, 11]. High SNR thresholds occur at about -6 dB, below which TDE performance rapidly degrades. Above the high SNR threshold, the ZZBs decrease linearly with increasing SNR, with a slope of about -0.5, similar to bandwidth limited pulse cases studied by Ziv and Zakai [2, 10]. The comparison between the ZZBs for FH and MC shows the two waveforms yield the same TDE performance conditioned on the same symbol rate. Moreover, the ZZBs track the MAP estimator threshold behavior and the MAP RMSE converges to the ZZB at high SNR as expected. At moderate SNR, the ZZBs bound the estimators tightly. At low SNR the MAP RMSE converges to $T/\sqrt{6}$, higher than the ZZB's convergence level. The gap can be quantitatively analyzed, similar to the pulsed waveform case [11].

6. CONCLUSIONS

We developed a Ziv-Zakai bound on time delay estimation for known frequency hopping or multicarrier waveforms over random wideband Gaussian channels. Correlation between frequencies is captured in the model. The ZZB reflects the penalty on TDE due to the unknown channel realization. Further topics of interest include comparison with other delay estimators, bias-variance tradeoffs, and the impact of unknown symbols.

7. APPENDIX: MGF OF A QUADRATIC FUNCTION OF A COMPLEX GAUSSIAN RANDOM VECTOR

Assume $\boldsymbol{r}$ is a complex circularly symmetric Gaussian vector with distribution $p(\boldsymbol{r})$ given by $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. For Hermitian symmetric matrices $\boldsymbol{\Psi}$, $\boldsymbol{g}$ and d, the MGF of $Q = \boldsymbol{r}^H\boldsymbol{\Psi}\boldsymbol{r} + 2\mathrm{Re}\{\boldsymbol{g}^H\boldsymbol{r}\} + d$ is defined as $\Theta(s) = E\{\exp(sQ)\}$. Using the same integration technique as shown in [12] for the quadratic form $\boldsymbol{r}^H\boldsymbol{\Psi}\boldsymbol{r}$, and after a similar manipulation as Theorem 3.2a.1 in [9], we can obtain a symmetric form of the MGF given by

$$\begin{aligned}\Theta(s) &= |\boldsymbol{I} - s\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\Sigma}^{\frac{1}{2}}|^{-1} \\ &\times \exp\Big\{s(\boldsymbol{\mu}^H\boldsymbol{\Psi}\boldsymbol{\mu} + 2\mathrm{Re}\{\boldsymbol{g}^H\boldsymbol{\mu}\} + d) + s^2(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{g} + \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\mu})^H \\ &\times (\boldsymbol{I} - s\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\Sigma}^{\frac{1}{2}})^{-1}(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{g} + \boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\mu})\Big\}. \qquad (25)\end{aligned}$$

If $\boldsymbol{g} = 0$ and $d = 0$, then Q shrinks to $Q = \boldsymbol{r}^H\boldsymbol{\Psi}\boldsymbol{r}$. Then, $\Theta(s)$ can be found as

$$\begin{aligned}\Theta(s) &= |\boldsymbol{I} - s\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\Sigma}^{\frac{1}{2}}|^{-1} \\ &\times \exp\Big\{s\boldsymbol{\mu}^H\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\Sigma}^{\frac{1}{2}})(\boldsymbol{I} - s\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{\Psi}\boldsymbol{\Sigma}^{\frac{1}{2}})^{-1}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu}\Big\}. \qquad (26)\end{aligned}$$

8. REFERENCES

[1] B. M. Sadler and R. J. Kozick, "A survey of time delay estimation performance bounds," in *Proc. IEEE Workshop on SAM*, July 2006, pp. 282–288.

[2] D. Chazan, M. Zakai, and J. Ziv, "Improved lower bounds on signal parameter estimation," *IEEE Trans. Inf. Theory*, vol. 21, no. 1, pp. 90–93, Jan. 1975.

[3] H. L. van Trees and K. L. Bell, *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking*, IEEE Press, 2007.

[4] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels: A Unified Approach to Performance Analysis*, John Wiley & Sons, New Jersey, 2nd edition, 2005.

[5] Z. Xu and B. M. Sadler, "Time delay estimation bounds in convolutive random channels," *IEEE Journal of Selected Topics in Signal Processing*, vol. 1, no. 3, pp. 418–430, Oct. 2007.

[6] R. J. Kozick and B. M. Sadler, "Frequency hopping waveform diversity for time delay estimation," in *Proc. 2006 Intl. Waveform Diversity and Design Conference*, Jan. 2006.

[7] R. J. Kozick and B. M. Sadler, "Bounds and algorithms for time delay estimation on parallel, flat fading channels," in *Proc. ICASSP*, Apr. 2008, pp. 2413–2416.

[8] William C. Jakes, Jr., Ed., *Microwave mobile communications*, chapter 1, pp. 46–51, John Wiley & Sons, 1974.

[9] A. M. Mathai and S. B. Provost, *Quadratic Forms in Random Variables: Theory and Applications*, Marcel Dekker, 1992.

[10] J. Ziv and M. Zakai, "Some lower bounds on signal parameter estimation," *IEEE Trans. Inf. Theory*, vol. IT-15, no. 3, pp. 386–391, May 1969.

[11] B. M. Sadler, N. Liu, and Z. Xu, "Ziv-zakai bounds on time delay estimation in unknown convolutive random channels," *IEEE Trans. Signal Process.*, 2009, in press.

[12] G. L. Turin, "The characteristic function of Hermitian quadratic forms in complex normal variables," *Biometrika*, vol. 47, no. 1/2, pp. 199–201, Jun. 1960.