A Bayesian Bound (ZZB) for Time Delay Estimation with Frequency Hopping or Multicarrier Transmission

Ning Liu

WIT Lab, Department of Electrical Engineering
University of California, Riverside

November 06, 2009

Outline

Outline

Definition and Purpose of TDE

- Definition:
  Estimate the signal propagation time t_0 from the corrupted and noisy received signal y.

  t_0: Generally random

  Channel: Known or unknown to receiver; Narrow or wideband

  Estimation: Bayesian techniques

- Applications of TDE
  - Geolocation
  - Synchonization and Timing Acquisation
  - Medical Imaging
  - Military Useness

Review of Bayesian Estimation

- MMSE is the conditional mean estimator

$$\widehat{\theta}_{\mathrm{MMSE}} \triangleq E_{t_0|y}\{t_0\} = \int t_0 p(t_0|y) d\theta$$

- ML estimator

$$\widehat{\theta}_{\mathrm{ML}} \triangleq \arg\max_{t_0} \{\ln p(y|t_0)\}$$

- MAP estimator is commonly used

$$\widehat{\theta}_{\mathrm{MAP}} \triangleq \arg\max_{t_0} \{\ln p(t_0|y)\}$$

The *a posteriori* probability is
$\ln p(t_0|y) = \ln p(y|t_0) + \ln p(t_0) - \ln p(y)$

Motivations

Why need a bound?

- Performance metric: Mean-Squared Error (MSE)

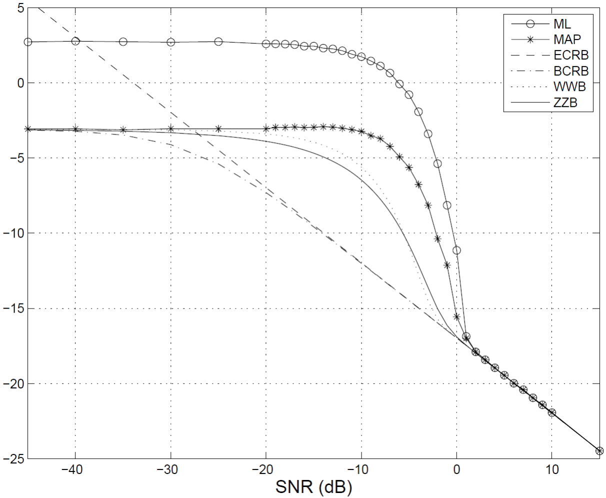
$$\bar{\epsilon}^2 = E_{y,t_0}\{[\widehat{t_0} - t_0]^2\} = E_{t_0}\left\{S(\widehat{t_0})\right\}$$

- Estimators have different MSE.
- Bounds predict the best possible MSE an estimator can achieve.

Why Ziv-Zakai Bound (ZZB) good?

- Bayesian bounds valid for random signal
- No restrictions estimators (unbiased, biased)
- Generally tighter than BCRB or average CRB

An example of ZZB and CRB



Bounds for frequency estimation. $p(\omega) = \frac{1}{2\pi\beta(a,a)}\left(\frac{\pi+\omega}{2\pi}\right)^{a-1}\left(\frac{\pi-\omega}{2\pi}\right)^{a+1}$.

Outline

Principle of ZZB: A Hypothesis Testing Problem

1. Conditional on a, two possible time delays:

   $H_0: \; t_0 = a,$ or $H_1: \; t_0 = a + \Delta,$ $\forall \, a, a+\Delta \in [0, T]$

2. A sub-optimal estimator produces $\widehat{t_0}$

   Decide H_0: $t_0 = a$ if $|\widehat{t_0} - a| < |\widehat{t_0} - a - \Delta|$

   Decide H_1: $t_0 = a + \Delta$ if $|\widehat{t_0} - a| > |\widehat{t_0} - a - \Delta|$

3. $P_e(a, a+\Delta)$: Minimum error probability by an optimum estimator

4. ZZB

$$\bar{\epsilon}^2 \geq \frac{1}{T} \int_0^{\Delta} \Delta \int_0^{T-\Delta} P_e(a, a+\Delta) da d\Delta$$

   If $P_e(a, a+\Delta) = P_e(\Delta)$, ZZB is

$$\bar{\epsilon}^2 \geq \frac{1}{T} \int_0^{\Delta} \Delta(T-\Delta) P_e(\Delta) d\Delta$$

Signal and Channel Model

- **Transmitted waveform**

$$s_i(t) = \sum_{k=-K_1}^{K_2} a_{i,k} p_{i,k}(t), \quad i = 1 \cdots N$$

FH: $p_{i,k}(t) = p(t - (i-1)MT_s - kT_s)$

MC: $p_{i,k}(t) = p_{\mathrm{N}}(t - kNT_s), \quad M = K_1 + K_2 + 1$

- **Channel Model**

$$g_i(t) = \sum_{l=1}^{L} \alpha_{i,l} \delta(t - (l-1)T_t)$$

$\boldsymbol{\alpha}_i = [\alpha_{i,1}, \cdots, \alpha_{i,L}]^T \sim \mathcal{CN}(\boldsymbol{\mu}_{\alpha i}, \boldsymbol{V}_i)$

$\boldsymbol{\alpha} = [\boldsymbol{\alpha}_1^T, \cdots, \boldsymbol{\alpha}_N^T]^T \sim \mathcal{CN}(\boldsymbol{\mu}_{\alpha}, \boldsymbol{V})$

Received Signal

- **Received Signal**

$$\begin{aligned} y_i(t) &= \sum_{l=1}^{L} \alpha_{i,l} s_i(t-(l-1)T_t - t_0) + n_i(t) \\ &= \boldsymbol{\alpha}_i^T \boldsymbol{s}_i(t-t_0) + n_i(t), \end{aligned}$$

$$\boldsymbol{s}_i(t-t_0) = [s_i(t-t_0), s_i(t-T_t-t_0), \cdots, s_i(t-(L-1)T_t-t_0)]^T$$

- **Received Signal for ZZB Development**

$$y_i(t) = \boldsymbol{\alpha}_i^T \boldsymbol{s}_{i,m} + n_i(t)$$

$\boldsymbol{s}_{i,m} = \boldsymbol{s}_i(t-m\Delta)$ Time delay t_0 is replaced by $m\Delta$ for ZZB development.

Distribution of Received Signal

- **Joint conditional pdf of** $\boldsymbol{y}(t) = [y_1(t), \cdots, y_N(t)]^T$

$$p(\boldsymbol{y}(t)|\boldsymbol{\alpha}, m\Delta) = \mathcal{K} \exp \sum_{i=1}^{N} \left[-\frac{1}{N_0} \int_{T_0} \left\| y_i(t) - \boldsymbol{\alpha}_i^T \boldsymbol{s}_{i,m} \right\|^2 dt \right]$$

- **Unconditional pdf by averaging over channel**

$$\begin{aligned} p(\boldsymbol{y}(t)|m\Delta) &= E_\alpha \left\{ p(\boldsymbol{y}(t)|\boldsymbol{\alpha}, m\Delta) \right\} \\ &\propto \exp \left\{ \boldsymbol{r}_m^H \boldsymbol{W} \boldsymbol{r}_m + 2\mathrm{Re}\{\boldsymbol{h}^H \boldsymbol{r}_m\} \right\} \end{aligned}$$

$\boldsymbol{r}_{i,m} \triangleq \int_{T_0} \boldsymbol{s}_{i,m}^* y(t) dt, \ \boldsymbol{r}_m = [\boldsymbol{r}_{1,m}^T, \cdots, \boldsymbol{r}_{N,m}^T]^T$

$\boldsymbol{W}$ and $\boldsymbol{h}$ depend on signal autocorrelation $\boldsymbol{S}_{00}$ and channel statistics $\boldsymbol{\mu}_\alpha$ and $\boldsymbol{V}$.

Log-likelihood Ratio Test

- **LLR to decide on H_0 and H_1**

$$\mathcal{L} \triangleq \ln \frac{p(\boldsymbol{y}(t)|0)}{p(\boldsymbol{y}(t)|\Delta)} = \boldsymbol{r}^H \boldsymbol{\Psi} \boldsymbol{r} + 2\mathrm{Re}\{\boldsymbol{g}^H \boldsymbol{r}\} \underset{H_1}{\overset{H_0}{\gtrless}} 0$$

$$\boldsymbol{r} = [\boldsymbol{r}_0^H \ \boldsymbol{r}_1^H]^H, \quad \boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{W} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{W} \end{bmatrix}, \quad \boldsymbol{g} = \begin{bmatrix} \boldsymbol{h} \\ -\boldsymbol{h} \end{bmatrix} = \boldsymbol{G}\boldsymbol{\mu}_\alpha.$$

- **pdf of LLR**

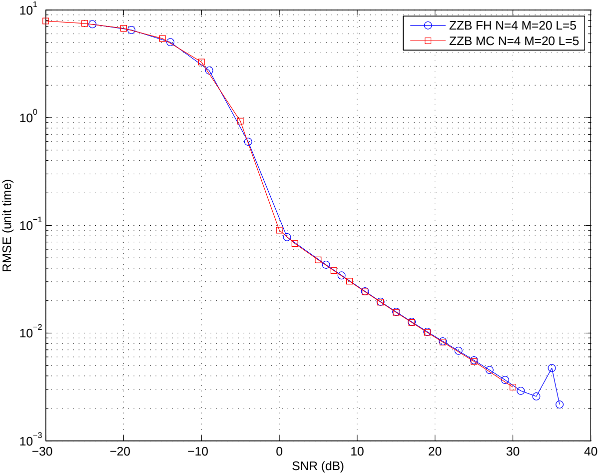
  pdf of $\boldsymbol{r}$ (Guassian) $\rightarrow$ MGF of $\mathcal{L}$ (Quadratic Gaussian) $\overset{\mathrm{FT}}{\rightarrow}$ pdf of $\mathcal{L}$

- **$P_e(\Delta)$ and ZZB**

$$P_e(\Delta) = \frac{1}{2} Pr\{\mathcal{L} < 0 | H_0\} + \frac{1}{2} Pr\{\mathcal{L} > 0 | H_1\}$$

$$\bar{\epsilon}^2 \geq \frac{1}{T} \int_0^{\Delta} \Delta(T - \Delta) P_e(\Delta) d\Delta$$

A Numerical Example



Outline

Special Cases

- Independent channels between frequencies
- Flat fading (L=1, Narrow band hops or subcarriers)
  - Flat fading (L=1) and Independent channels
- Known (Deterministic) Channel

Special Cases

- Independent channels between frequencies
- Flat fading (L=1, Narrow band hops or subcarriers)
  - Flat fading (L=1) and Independent channels
- Known (Deterministic) Channel

Closed Form ZZB for the Special Case

- **LLR**

$$\mathcal{L} = \boldsymbol{r}^H \boldsymbol{\Psi} \boldsymbol{r} + 2\mathrm{Re}\{\boldsymbol{g}^H \boldsymbol{r}\} = \sum_{i=1}^{N} W_i (r_{i,0}^2 - r_{i,1}^2) + 2\mathrm{Re}\{h_i^* (r_{i,0} - r_{i,1})\}.$$

- **Closed Form** of P_e

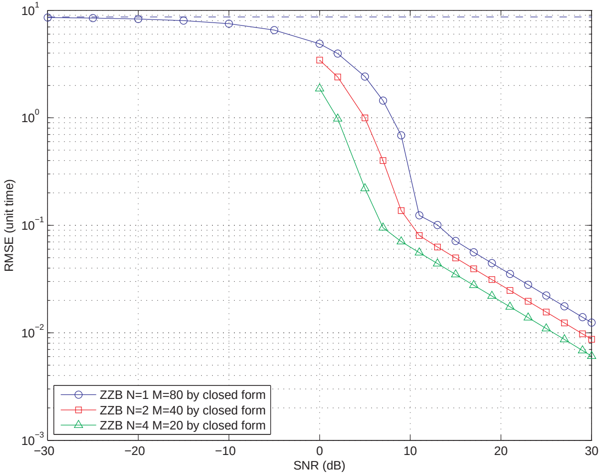
$$P_e(\Delta) = \Pr\{\mathcal{L}|H_0 < 0\} = \Pr\left\{\sum_{i=1}^{N}\left[\left|r_{i,0}|H_0 + \frac{h_i}{W_i}\right|^2 - \left|r_{i,1}|H_0 + \frac{h_i}{W_i}\right|^2\right] < 0\right\}$$

$$= Q_1(a,b) - \left[1 - \frac{\sum_{i=0}^{N-1}\binom{2N-1}{i}\eta^i}{(1+\eta)^{2N-1}}\right]\exp\left(-\frac{a^2+b^2}{2}\right) I_0(ab)$$

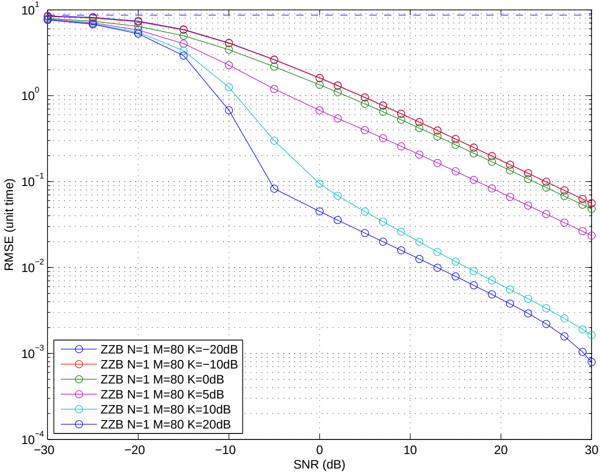
$$+ \frac{1}{(1+\eta)^{2N-1}}\left\{\sum_{i=2}^{N}\binom{2N-1}{N-i}\left\{\eta^{N-i}[Q_i(a,b) - Q_1(a,b)]\right.\right.$$

$$\left.\left. - \eta^{N-1+i}[Q_i(b,a) - Q_1(b,a)]\right\}\right\}$$

A Numerical Example



Numerical Examples of Special Case



Outline

Summary

- Developed a Bayesian bound (ZZB) for TDE in unknown convolutive random channel, providing a general theoretical framework for both FH and MC signal.
- The ZZB is valid for both wideband and narrow band channels, both FFH and SFH, both LOS and NLOS channels, different channel correlation profiles, and various pulse shaping and waveforms.
- FH or MC transmission provides TDE diversity under frequency-selective channels.
- No closed form expression of ZZB for the general case. Properties of Hermitian matrices used for numerical evaluation.
- Closed form expression exists for some special cases.

Thank you!